

AN EXPLICIT FORMULA FOR COMPUTING BERNOULLI NUMBERS OF THE SECOND KIND IN TERMS OF STIRLING NUMBERS OF THE FIRST KIND

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ABSTRACT. In the paper, the author finds an explicit formula for computing Bernoulli numbers of the second kind in terms of Stirling numbers of the first kind.

1. INTRODUCTION

It is well known that Stirling numbers of the first kind $s(n, k)$ for $n \geq k \geq 1$ may be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{n!}, \quad |x| < 1 \quad (1.1)$$

and that Bernoulli numbers of the second kind b_n for $n \geq 0$ may be generated by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} b_n x^n. \quad (1.2)$$

In [5], the following formula for computing Bernoulli numbers of the second kind in terms of Stirling numbers of the first kind was derived:

$$b_n = \frac{1}{n!} \sum_{k=0}^n \frac{s(n, k)}{k+1}. \quad (1.3)$$

The main aim of this paper is to find a new and explicit formula for computing Bernoulli numbers of the second kind in terms of Stirling numbers of the first kind. The main result of this paper may be stated as the following theorem.

Theorem 1.1. *For $n \geq 2$, Bernoulli numbers of the second kind b_n may be computed in terms of Stirling numbers of the first kind $s(n, k)$ by*

$$b_n = \frac{1}{n!} \sum_{k=1}^{n-1} (-1)^k \frac{s(n-1, k)}{(k+1)(k+2)}. \quad (1.4)$$

As a remark, a relation between the harmonic numbers and Stirling numbers of the first kind $s(n, 2)$ is also derived.

2. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is based on some results elementarily and inductively obtained in [6] and its preprint [7]. These results can be recited as follows.

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- (1) Corollary 2.3 in [6, 7] states that Stirling numbers of the first kind $s(n, k)$ for $1 \leq k \leq n$ may be computed by

$$s(n, k) = (-1)^{n+k} (n-1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-2}=1}^{\ell_{k-3}-1} \frac{1}{\ell_{k-2}} \sum_{\ell_{k-1}=1}^{\ell_{k-2}-1} \frac{1}{\ell_{k-1}}. \quad (2.1)$$

This formula may be reformulated as

$$(-1)^{n-k} \frac{s(n, k)}{(n-1)!} = \sum_{m=k-1}^{n-1} \frac{1}{m} \left[(-1)^{m-(k-1)} \frac{s(m, k-1)}{(m-1)!} \right]. \quad (2.2)$$

- (2) Corollary 2.4 in [6, 7] reads that for $1 \leq k \leq n$ Stirling numbers of the first kind $s(n, k)$ satisfies the recursion

$$s(n+1, k) = s(n, k-1) - ns(n, k). \quad (2.3)$$

This is a recovery of the triangular relation for $s(n, k)$.

- (3) Theorem 3.1 in [6, 7] tells that Bernoulli numbers of the second kind b_n for $n \geq 2$ may be computed by

$$b_n = (-1)^n \frac{1}{n!} \left(\frac{1}{n+1} + \sum_{k=2}^n \frac{a_{n,k} - na_{n-1,k}}{k!} \right), \quad (2.4)$$

where

$$a_{n,2} = (n-1)! \quad (2.5)$$

and, for $n+1 \geq k \geq 3$,

$$a_{n,k} = (k-1)!(n-1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-3}=1}^{\ell_{k-4}-1} \frac{1}{\ell_{k-3}} \sum_{\ell_{k-2}=1}^{\ell_{k-3}-1} \frac{1}{\ell_{k-2}}. \quad (2.6)$$

Observing the expressions (2.1) and (2.6), we obtain

$$a_{n,k} = (-1)^{n+k-1} (k-1)! s(n, k-1), \quad n+1 \geq k \geq 2. \quad (2.7)$$

See [6, (2.18)] and [8, (6.7)]. By this and the recursion (2.3), it follows that

$$\begin{aligned} a_{n,k} - na_{n-1,k} &= (-1)^{n+k-1} (k-1)! [s(n, k-1) + ns(n-1, k-1)] \\ &= (-1)^{n+k-1} (k-1)! [s(n-1, k-1) + s(n-1, k-2)]. \end{aligned}$$

Substituting this into (2.4) reveals that

$$\begin{aligned} b_n &= \frac{(-1)^n}{n!} \left(\frac{1}{n+1} + \sum_{k=2}^n \frac{(-1)^{n+k-1} [s(n-1, k-1) + s(n-1, k-2)]}{k} \right) \\ &= \frac{(-1)^n}{(n+1)!} + \frac{1}{n!} \sum_{k=2}^n \frac{(-1)^{k-1} [s(n-1, k-1) + s(n-1, k-2)]}{k} \\ &= \frac{(-1)^n}{(n+1)!} + \frac{1}{n!} \left[\sum_{k=2}^n \frac{(-1)^{k-1} s(n-1, k-1)}{k} + \sum_{k=2}^n \frac{(-1)^{k-1} s(n-1, k-2)}{k} \right] \\ &= \frac{(-1)^n}{(n+1)!} + \frac{1}{n!} \left[\sum_{k=2}^n \frac{(-1)^{k-1} s(n-1, k-1)}{k} + \sum_{k=1}^{n-1} \frac{(-1)^k s(n-1, k-1)}{k+1} \right] \\ &= \frac{(-1)^n}{(n+1)!} + \frac{1}{n!} \frac{(-1)^{n-1}}{n} + \frac{1}{n!} \sum_{k=2}^{n-1} (-1)^{k-1} s(n-1, k-1) \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \frac{1}{n!} (-1)^{n-1} \left(\frac{1}{n} - \frac{1}{n+1} \right) + \frac{1}{n!} \sum_{k=2}^{n-1} (-1)^{k-1} s(n-1, k-1) \left(\frac{1}{k} - \frac{1}{k+1} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n!} \sum_{k=2}^n (-1)^{k-1} s(n-1, k-1) \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= \frac{1}{n!} \sum_{k=2}^n (-1)^{k-1} \frac{s(n-1, k-1)}{k(k+1)}.
\end{aligned}$$

Notice that in the above argument, we use the convention $s(n, 0) = 0$ for $n \in \mathbb{N}$ and the fact $s(n, n) = 1$ for $n \geq 0$. The proof of Theorem 1.1 is complete.

3. REMARKS

In this section, we show some new findings by several remarks.

Remark 3.1. The idea in Theorem 1.1 and its proof ever implicitly thrilled through in [8, Remark 6.7].

Remark 3.2. Making use of the relation (2.7) in [6, Theorem 2.1] leads to

$$\left(\frac{1}{\ln x} \right)^{(n)} = \frac{1}{x^n} \sum_{k=1}^n (-1)^k k! s(n, k) \left(\frac{1}{\ln x} \right)^{k+1}, \quad n \in \mathbb{N}. \quad (3.1)$$

This recovers the first formula in [4, Lemma 2].

By the way, the formulas (3.4) and (3.5) in [6, Corollary 3.1] recover the second formula in [4, Lemma 2].

Remark 3.3. In [6, Remark 2.2], it was conjectured that the sequence $a_{n,k}$ for $n \in \mathbb{N}$ and $2 \leq k \leq n+1$ is increasing with respect to n while it is unimodal with respect to k for given $n \geq 4$. This conjecture may be partially confirmed as follows.

From (2.6), the increasing monotonicity of the sequence $a_{n,k}$ with respect to n follows straightforwardly.

It is clear that the sequence $(k-1)!$ is increasing with k and the sequence

$$\sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-3}=1}^{\ell_{k-4}-1} \frac{1}{\ell_{k-3}} \sum_{\ell_{k-2}=1}^{\ell_{k-3}-1} \frac{1}{\ell_{k-2}}$$

is decreasing with k . Since $a_{n,n+1} = n!$, see the equation (2.5) or [6, (2.8)], we obtain that

$$a_{n,2} < a_{n,n+1}, \quad n \geq 2. \quad (3.2)$$

In [8, Theorem 2.1], the integral representation

$$s(n, k) = \binom{n}{k} \lim_{x \rightarrow 0} \frac{d^{n-k}}{dx^{n-k}} \left\{ \left[\int_0^\infty \left(\int_{1/e}^1 t^{xu-1} dt \right) e^{-u} du \right]^k \right\} \quad (3.3)$$

was created for $1 \leq k \leq n$. Hence,

$$\begin{aligned}
s(n, n-1) &= n \lim_{x \rightarrow 0} \frac{d}{dx} \left\{ \left[\int_0^\infty \left(\int_{1/e}^1 t^{xu-1} dt \right) e^{-u} du \right]^{n-1} \right\} \\
&= n(n-1) \lim_{x \rightarrow 0} \left[\int_0^\infty \left(\int_{1/e}^1 t^{xu-1} dt \right) e^{-u} du \right]^{n-2} \\
&\quad \times \lim_{x \rightarrow 0} \left[\int_0^\infty \left(\int_{1/e}^1 t^{xu-1} \ln t dt \right) u e^{-u} du \right] \\
&= n(n-1) \left[\int_0^\infty \left(\int_{1/e}^1 \frac{1}{t} dt \right) e^{-u} du \right]^{n-2} \int_0^\infty \left(\int_{1/e}^1 \frac{\ln t}{t} dt \right) u e^{-u} du \\
&= -\frac{1}{2} n(n-1).
\end{aligned}$$

As a result, by (2.7), it follows that

$$a_{n,n} = -(n-1)!s(n, n-1) = \frac{n-1}{2}n! \geq a_{n,n+1}, \quad n \geq 3. \quad (3.4)$$

Combining this with (3.2) shows that the sequence $a_{n,k}$ for given $n \geq 4$ has at least one maximum with respect to $2 < k < n+1$.

Remark 3.4. By the integral representation (3.3) and direct computation, we can recover that

$$\begin{aligned} s(n, 1) &= \binom{n}{1} \lim_{x \rightarrow 0} \frac{d^{n-1}}{dx^{n-1}} \int_0^\infty \left(\int_{1/e}^1 t^{xu-1} dt \right) e^{-u} du \\ &= n \lim_{x \rightarrow 0} \int_0^\infty \left[\int_{1/e}^1 t^{xu-1} (\ln t)^{n-1} dt \right] u^{n-1} e^{-u} du \\ &= n \int_0^\infty \left[\int_{1/e}^1 \frac{(\ln t)^{n-1}}{t} dt \right] u^{n-1} e^{-u} du \\ &= (-1)^{n+1} \int_0^\infty u^{n-1} e^{-u} du \\ &= (-1)^{n+1} (n-1)! \end{aligned}$$

and

$$\begin{aligned} s(n, 2) &= \binom{n}{2} \lim_{x \rightarrow 0} \frac{d^{n-2}}{dx^{n-2}} \left\{ \left[\int_0^\infty \left(\int_{1/e}^1 t^{xu-1} dt \right) e^{-u} du \right]^2 \right\} \\ &= \binom{n}{2} \lim_{x \rightarrow 0} \sum_{k=0}^{n-2} \binom{n-2}{k} \int_0^\infty \left[\int_{1/e}^1 t^{xu-1} (\ln t)^k dt \right] u^k e^{-u} du \\ &\quad \times \int_0^\infty \left[\int_{1/e}^1 t^{xu-1} (\ln t)^{n-k-2} dt \right] u^{n-k-2} e^{-u} du \\ &= \binom{n}{2} \sum_{k=0}^{n-2} \binom{n-2}{k} \int_0^\infty \left[\int_{1/e}^1 \frac{(\ln t)^k}{t} dt \right] u^k e^{-u} du \\ &\quad \times \int_0^\infty \left[\int_{1/e}^1 \frac{(\ln t)^{n-k-2}}{t} dt \right] u^{n-k-2} e^{-u} du \\ &= (-1)^n \binom{n}{2} \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{k!}{k+1} \frac{(n-k-2)!}{n-k-1} \\ &= (-1)^n (n-2)! \binom{n}{2} \sum_{k=0}^{n-2} \frac{1}{(k+1)(n-k-1)} \\ &= (-1)^n \frac{n!}{2} \sum_{k=0}^{n-2} \frac{1}{(k+1)(n-k-1)} \\ &= (-1)^n \frac{(n-1)!}{2} \sum_{k=0}^{n-2} \left(\frac{1}{k+1} + \frac{1}{n-k-1} \right) \\ &= (-1)^n (n-1)! H(n-1), \end{aligned}$$

where

$$H(n) = \sum_{k=1}^n \frac{1}{k} \quad (3.5)$$

is the n -th harmonic number. Consequently, we find a relation

$$s(n, 2) = (-1)^n (n-1)! H(n-1), \quad n \in \mathbb{N} \quad (3.6)$$

or, equivalently,

$$H(n) = \frac{(-1)^{n+1}s(n+1, 2)}{n!}, \quad n \in \mathbb{N} \quad (3.7)$$

between the n -th harmonic number $H(n)$ and Stirling numbers of the first kind $s(n, 2)$.

The relations (3.6) and (3.7) may also be deduced by considering (2.6) and (2.7) and may also be found in [1, p. 275, (6.58)].

For more information on the n -th harmonic numbers $H(n)$, please refer to [2] and closely related references therein.

Remark 3.5. For more information on the second Stirling numbers and the first kind Bernoulli numbers, please refer to [3, 8] and closely related references therein.

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